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# Evolution Equations in Banach Space with Variable Domain

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1. Let $E$ be a separable reflexive Banach space over $\mathbf{C}$, the complex numbers. We assume that $E$ may be identified with a dense subspace of $E'$, the antidual of $E$, and that the injection $E \to E'$ is continuous. Let the norm in $E$ be $|f|$, $f \in E$. We suppose also that there is a constant $k > 0$ such that for $f \in E$, $\langle f, f \rangle_{E,E'} \geqslant k \, |f|_{E'}^2$.

Next we let $[0, T]$ be a finite real interval and for each $t \in [0, T]$ we suppose that $V(t)$ is a dense subspace of $E$, having its own Banach space structure, with norm $\| u \|_t$, $u \in V(t)$. We suppose in addition that there is a constant $c > 0$ such that $\| u \|_t \geqslant c \, | u |$ for all $u \in V(t)$, $0 \leqslant t \leqslant T$. It follows that we may make the identifications

$$V(t) \subset E \subset E' \subset V'(t).$$

Assuming that $V(t)$ varies measurably, we let $W$ be the space of (classes of) functions

$$u \in L^p(0, T; E) : u(t) \in V(t) \text{ a.e.} \qquad \text{and} \qquad \int_0^T \| u \|_t^p \, dt < \infty.$$

We shall assume $p \geqslant 2$ and $1/p + 1/q = 1$ so that $W$ is a Banach space and

$$W \subset L^p(0, T; E) \subset L^q(0, T; E') \subset W'.$$

Let $A : W \to W'$ be a possibly nonlinear operator, and let $L_s$ be the closure in $W \times W'$ (hypotheses guaranteeing closability will be given later) of $d/dt$ with domain

$$u \in W : \frac{du}{dt} \in L^q(0, T; E') \qquad \text{and} \qquad u(0) = 0.$$

Derivatives are taken in the sense of distributions on $(0, T)$ with values in $E'$.

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We then consider the abstract evolution equation

$$L_s u + Au = f, \qquad u \in D(L_s),$$

where $f$ is a given element of $W'$. Our main theorem, which extends Refs. [9, 10] to the Banach space case, and which partially extends Ref. [5] to the case of variable domain, is found in Section 3. It yields the existence and uniqueness of solutions of this equation. To obtain this result, we make the following hypotheses on $V(t)$:

$V(t)$ is the domain of $S(t)$ with the graph norm, where $S(t)$ is an unbounded, closed, linear operator in $E$ satisfying

(1.1) For $u, v \in D(S^2(t))$, $\langle S^2(t)u, v\rangle_{E,E'} = \langle u, S^2(t)v\rangle_{E,E'}$.

(1.2) For $0 \leqslant t \leqslant T$, $S^2(t)$ is invertible in $\mathscr{L}(E)$ and is of type $(\pi/2, c_1)$ where $c_1$ is a constant independent of $t \in [0, T]$, i.e., the resolvant set of $-S^2(t)$ contains the sector $|\arg \lambda| < \pi/2$, $\|\lambda(\lambda + S^2(t))\|_{\mathscr{L}(E)}$ is uniformly bounded in each sector $|\arg\lambda| \leqslant \pi/2 - \epsilon$, and $\|\lambda(\lambda + S^2(t))\|_{\mathscr{L}(E)} \leqslant c_1$ for $\lambda > 0$.

(1.3) For each $f \in E$, $g \in E'$, the function $t \to \langle S^{-2}(t)f, g\rangle_{E,E'}$ is continuously differentiable on $[0, T]$.

Under the hypotheses (1.1), (1.2), (1.3) we are able to prove that the operator $du/dt$ defined above is closable, and that if $A$ is hemicontinuous, bounded, coercive, and strictly monotone, then

$$L_s + A : D(L_s) \to W'$$

is a bijection.

Section 2 is devoted to preliminary consequences of (1.1), (1.2), and (1.3), while Section 4 considers examples from partial differential equations.

**2.** As in Section 1, let $E$ be a separable reflexive Banach space over $\mathbf{C}$, with $E \subset E'$, where $E'$ is the space of continuous antilinear forms on $E$ with the usual dual norm. We denote the norm by $|f|$ for $f \in E$, and we suppose that there is a constant $k > 0$ such that for $f \in E$, $\langle f, f\rangle_{E,E'} \geqslant k|f|_{E'}^2$.

Now for $0 \leqslant t \leqslant T$, let $S(t)$ be a family of closed, densely defined, linear operators in $E$, satisfying (1.1), (1.2), and (1.3). We let $V(t)$ denote the domain of $S(t)$ equipped with the graph norm. Because of (1.2), the norm in $V(t)$, $\|u\|_t$, is equivalent to $|S(t)u|$ for all $u \in V(t)$.

We shall first examine some consequences of (1.1).

LEMMA 2.1. *If $S^2(t)$ satisfies* (1.1), *then $S(t)$ and (for $\lambda \geqslant 0$) $M_\lambda(t) = (\lambda + S^2(t))^{-1}$ both satisfy* (1.1).

*Proof.* (1.1) implies that $S^{-2}(t)$ is also "symmetric" in the sense of (1.1). By (1.2), we may expand $M_\lambda(t)$ in a power series

$$M_\lambda(t) = S^{-2}(t) \sum_{n=0}^{\infty} (-1)^n (\lambda S^{-2}(t))^n$$

which is convergent in $\mathscr{L}(E)$ for $0 \leqslant \lambda < \| S^{-2}(t)\|^{-1}$. It follows that $M_\lambda(t)$ is "symmetric" in the sense of (1.1) for $0 \leqslant \lambda < \| S^{-2}(t)\|^{-1}$. Thus for $f$ and $g \in E$, the functions

$$\lambda \to \langle M_\lambda(t) f, g\rangle_{E,E'} \qquad \text{and} \qquad \langle f, M_\lambda(t)\, g\rangle_{E,E'}$$

are both real analytic on $\lambda \geqslant 0$, and they agree for $0 \leqslant \lambda < \| S^{-2}(t)\|^{-1}$. Hence they must agree for all $\lambda \geqslant 0$ and it follows that $M_\lambda(t)$ satisfies (1.1) for all $\lambda \geqslant 0$. From the integral representation (cf. Ref. [13])

$$S^{-1}(t) = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} M_\lambda(t)\, d\lambda,$$

the integral being norm-convergent in $\mathscr{L}(E)$, we deduce that $S^{-1}(t)$, and consequently $S(t)$ satisfies (1.1). Q.E.D.

In general, if $\Gamma : E \to E$ is a bounded linear operator, we shall denote by $\Gamma^* : E' \to E'$ the adjoint. Now if $\Gamma$ satisfies (1.1), then it follows that $\Gamma^*$ is an extension of $\Gamma$ to $E'$, and that

$$\| \Gamma^* \|_{\mathscr{L}(E')} = \| \Gamma \|_{\mathscr{L}(E)} \,.$$

Since $S^{-1}(t) : E \to V(t)$ continuously, we may define an adjoint linear operator

$$(S^{-1}(t))^\wedge : V'(t) \to E'.$$

It may be shown, by virtue of the "symmetry" of $S^{-1}(t)$, that $(S^{-1}(t))^\wedge$ is an extension to $V'(t)$ of $S^{-1}(t)^*$, which is of course an extension of $S^{-1}(t)$.

We also note that for $f \in E'$,

$$| f |_{V'(t)} = | S^{-1}(t)^* f |_{E'} \,. \tag{2.1}$$

In fact,

$$\begin{aligned} | f |_{V'(t)} &= \sup_{\|v\|_t \leqslant 1} |\langle f, v\rangle|_{V'(t),V(t)} \\ &= \sup_{|g|_E \leqslant 1} |\langle f, S^{-1}(t) g\rangle|_{E',E} = | S^{-1}(t)^* f |_{E'} \,. \end{aligned}$$

Next let us consider the smoothness condition (1.3).

LEMMA 2.2. *Assume that* $S^2(t)$ *satisfies* (1.1), (1.2), *and* (1.3). *Then there is a bounded linear operator* $S^{-2}(t)^{\cdot} \in \mathscr{L}(E)$ *which is "symmetric" in the sense of* (1.1) *such that for* $f \in E$ *and* $g \in E'$,

$$\frac{d}{dt}\langle S^{-2}(t) f, g\rangle_{E,E'} = \langle S^{-2}(t)^{\cdot} f, g\rangle_{E,E'}\,.$$

*Furthermore, there is a constant* $c_2 > 0$ *such that*

$$\| S^{-2}(t)^{\cdot} \|_{\mathscr{L}(E)} \leqslant c_2\,, \qquad 0 \leqslant t \leqslant T.$$

*Proof.* Since $\lim_{h\to 0}\langle([S^{-2}(t+h) - S^{-2}(t)]/h) f, g\rangle_{E,E'}$ exists for each $f \in E$, $g \in E'$, the quotients $[S^{-2}(t+h) - S^{-2}(t)]/h$ are weakly bounded, hence strongly bounded in $E$. Using the reflexivity of $E$, we may deduce that there is an element $S^{-2}(t)^{\cdot} f \in E$ such that

$$\lim_{h\to 0}\left\langle \frac{S^{-2}(t+h) - S^{-2}(t)}{h} f, g\right\rangle_{E,E'} = \langle S^{-2}(t)^{\cdot} f, g\rangle_{E,E'}\,.$$

Similarly, there is an element $(S^{-2}(t)^{*})^{\cdot} g \in E'$ such that

$$\begin{aligned}
\lim_{h\to 0}&\left\langle \frac{S^{-2}(t+h) - S^{-2}(t)}{h} f, g\right\rangle_{E,E'} \\
&= \lim_{h\to 0}\left\langle f, \frac{S^{-2}(t+h)^{*} - S^{-2}(t)^{*}}{h} g\right\rangle_{E,E'} \\
&= \langle f, (S^{-2}(t)^{*})^{\cdot} g\rangle_{E,E'}\,.
\end{aligned}$$

Thus for $f \in E$ with $| f | \leqslant 1$, we have

$$|\langle S^{-2}(t)^{\cdot} f, g\rangle| \leqslant |(S^{-2}(t)^{*})^{\cdot} g|_{E'}\,,$$

and it follows that $S^{-2}(t)^{\cdot}$ is a bounded operator on $E$. Furthermore, from the continuity of $\langle S^{-2}(t)^{\cdot} f, g\rangle$ on $[0, T]$, we have that for each $f \in E$, $t \to S^{-2}(t)^{\cdot} f$ is weakly bounded, hence strongly bounded. Then by an application of the Banach–Steinhaus theorem, we have that $S^{-2}(t)^{\cdot}$ is bounded in $\mathscr{L}(E)$. The symmetry of $S^{-2}(t)$ is an immediate consequence of the symmetry of the quotients $[S^{-2}(t+h) - S^{-2}(t)]/h$. Q.E.D.

COROLLARY. $S^{-2}(t)$, $S^{-1}(t)$, *and* $M_\lambda(t) = (\lambda + S^2(t))$ $(\lambda \geqslant 0)$ *are all continuous in* $\mathscr{L}(E)$ *on* $[0, T]$.

*Proof.* For $f \in E$, $g \in E'$,

$$\begin{aligned}
\langle [S^{-2}(t) - S^{-2}(s)] f, g\rangle &= \int_s^t \langle S^{-2}(\xi)^{\cdot} f, g\rangle\, d\xi \\
&\leqslant c_2(t - s)\, | f |_E\, | g |_{E'}\,.
\end{aligned}$$

Thus $S^{-2}(t)$ is even Lipschitz continuous in norm.

We may write, for $\lambda \geqslant 0$,

$$M_\lambda(t) = S^{-2}(t)(I + \lambda S^{-2}(t))^{-1}.$$

Since the map $A \to A^{-1}$ is continuous in the set of invertible elements in $\mathscr{L}(E)$, it follows that $t \to M_\lambda(t)$ is continuous in $\mathscr{L}(E)$.

Finally, we use the representation formula

$$S^{-1}(t) = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} M_\lambda(t)\, d\lambda,$$

hypothesis (1.2), and the Lebesgue dominated convergence theorem to deduce that $t \to S^{-1}(t)$ is also continuous in $\mathscr{L}(E)$. Q.E.D.

LEMMA 2.3. *If $S^2(t)$ satisfies* (1.1), (1.2), *and* (1.3), *then for $\lambda \geqslant 0$, $M_\lambda(t)$ also satisfies* (1.3). *Furthermore, $M_\lambda(t)^\cdot$ is a bounded operator in $\mathscr{L}(E)$, symmetric in the sense of* (1.1) *and*

$$M_\lambda(t)^\cdot = -S^2(t)\, M_\lambda(t)\, S^{-2}(t)^\cdot\, S^2(t)\, M_\lambda(t). \tag{2.2}$$

*Proof.* $S^2(t)\, M_\lambda(t) = (I + \lambda S^{-2}(t))^{-1}$ is continuous in $\mathscr{L}(E)$, by the same argument as in the corollary to Lemma 2.2. Thus $(S^2(t)\, M_\lambda(t))^*$ is continuous in $\mathscr{L}(E')$. Now the appropriate differences are

$$M_\lambda(t) - M_\lambda(s) = S^2(s)\, M_\lambda(s)[S^{-2}(s) - S^{-2}(t)]\, S^2(t)\, M_\lambda(t)$$

and

$$\frac{1}{t-s} \langle [M_\lambda(t) - M_\lambda(s)] f, g\rangle_{E,E'}$$
$$= \frac{1}{t-s} \langle [S^{-2}(s) - S^{-2}(t)]\, S^2(t)\, M_\lambda(t) f, (S^2(s)\, M_\lambda(s))^* g\rangle_{E,E'} .$$

Taking the limit as $t \to s$ we obtain (2.2). Q.E.D.

LEMMA 2.4. *Suppose that* (1.1), (1.2), *and* (1.3) *are satisfied. Then there is a constant $c > 0$, independent of $t$, such that for $v \in V(t) = D(S(t))$,*

$$| \lambda S^{-1}(t)\, M_\lambda(t)^\cdot v \mid \leqslant c \mid S(t) v \mid$$

*and $\lambda S^{-1}(t)\, M_\lambda(t)^\cdot v \to 0$ in $L^p(0, T; E)$ as $\lambda \to +\infty$, $1 \leqslant p - \infty$.*

*Proof.* First we must estimate the norm of $S(t)\, M_\lambda(t) = S(t)(\lambda + S^2(t))^{-1}$ for $\lambda > 0$. From (1.2) and a familiar argument with contour integrals, we may deduce that

$$S(t)(\lambda + S^2(t))^{-1} = \frac{1}{\pi} \int_0^\infty \frac{\mu^{1/2}}{\lambda - \mu} (\mu + S^2(t))^{-1}\, d\mu.$$

A change of variable in this representation and (1.2) yield the estimate

$$\| S(t)(\lambda + S^2(t))^{-1} \|_{\mathscr{L}(E)} \leqslant c_1 \lambda^{-1/2} \qquad \text{for} \quad \lambda > 0,$$

where $c_1$ is a constant independent of $t$.

Now let $v \in V(t) = D(S(t))$. Then, using Lemma 2.4,

$$\begin{aligned} | \lambda S^{-1}(t) M_\lambda(t) v |_E &\leqslant \lambda \| S(t) M_\lambda(t) \|^2 \| S^{-2}(t) \| \, | S(t) v | \\ &\leqslant c_1 ( \sup_{0 \leqslant t \leqslant T} \| S^{-2}(t) \| ) \, | S(t) v |. \end{aligned}$$

Next suppose that $v \in D(S^2(t))$. In this case

$$\begin{aligned} | \lambda S^{-1}(t) M_\lambda(t) v |_E &\leqslant \| S(t) M_\lambda(t) \| \, \| S^{-2}(t) \| \, \| \lambda M_\lambda(t) \| \, | S^2(t) v | \\ &\leqslant c_2 \lambda^{-1/2} ( \sup_{0 \leqslant t \leqslant T} \| S^{-2}(t) \| ) \, | S^2(t) v | \end{aligned}$$

and thus $\lambda S^{-1}(t) M_\lambda(t) v \to 0$ as $\lambda \to +\infty$. Since $D(S^2(t))$ is dense in $D(S(t))$, it follows that

$$\lambda S^{-1}(t) M_\lambda(t) v \to 0 \quad \text{in} \quad E \quad \text{as} \quad \lambda \to +\infty \quad \text{for all} \quad v \in D(S(t)). \qquad \text{Q.E.D.}$$

3. We assume that (1.1), (1.2), and (1.3) are satisfied, and we let $V(t)$ be the domain $D(S(t))$ with the graph norm. It follows that we make the identifications

$$\begin{gathered} V(t) \subset E \subset E' \subset V'(t), \\ \langle u, v \rangle_{V(t), V'(t)} = \langle u, v \rangle_{E, E'} \qquad \text{for} \quad u \in V(t), \quad v \in E'. \end{gathered} \tag{3.1}$$

Since $t \to S^{-1}(t)$ is continuous in $\mathscr{L}(E)$, we must have that

$$t \to \langle S^{-1}(t) f, g \rangle_{E, E'}$$

is measurable for $f \in E$, $g \in E'$, and that there is a constant $c > 0$ such that

$$\| u \|_t = | S(t) u | \geqslant c \, | u |, \qquad u \in V(t),$$

for $0 \leqslant t \leqslant T$. Thus we may define $W$ as the space of (classes of) functions

$$u \in L^p(0, T; E) : u(t) \in V(t) \text{ a.e.} \qquad \text{and} \qquad \int_0^T \| u \|_t^p \, dt < \infty.$$

We shall assume $1 < p < \infty$. Because $S(t)$ is a closed operator and $E$ is reflexive, $W$ is a reflexive Banach space for the obvious norm. In general we shall let $q$ denote the conjugate index to $p$, i.e., $1/p + 1/q = 1$. We may identify $W'$ with

$$L^q(0, T; V'(t))$$

and if $p \geqslant 2$ we can further identify

$$W \subset L^p(0, T; E) \subset L^q(0, T; E') \subset W'.$$

Now we define the operators $L$ and $L'$ by

$$Lu = \frac{du}{dt} \quad \text{and} \quad D(L) = \left\{ u \in W : \frac{du}{dt} \in L^q(0, T; E'), u(0) = 0 \right\},$$

$$L'u = -\frac{du}{dt} \quad \text{and} \quad D(L') = \left\{ u \in W : \frac{du}{dt} \in L^q(0, T; E'), u(T) = 0 \right\}.$$

Derivatives are taken in the sense of distributions on $(0, T)$ with values in $E'$.

LEMMA 3.1. *Assume that* (1.1), (1.2), *and* (1.3) *are satisfied and* $1 < p < \infty$. *Then* $D(L)$ *and* $D(L')$ *are dense in* $W$.

*Proof.* Let $\varphi \in C_0^\infty(R)$ with support contained in $[0, 1]$, $\varphi \geqslant 0$, and $\int \varphi \, dt = 1$. Set $\varphi_n(t) = n\varphi(nt)$. For $u \in W$, we shall approximate by

$$v_{\lambda,n} = \lambda M_\lambda(u * \varphi_n), \quad \lambda \geqslant 0, \qquad n = 1, 2, 3, \ldots .$$

Since $\| S(t) M_\lambda(t) \|_{\mathscr{L}(E)} \leqslant c'$ by (1.2), and $u * \varphi_n \in L^p(0, T; E)$, it follows that $v_{\lambda,n} \in W$. Now

$$\frac{d}{dt}(v_{\lambda,n}) = \lambda M_\lambda^{\cdot}(u * \varphi_n) + \lambda M_\lambda \left( u * \frac{d\varphi_n}{dt} \right),$$

and it follows from Lemma 2.3 that $\| M_\lambda(t)^{\cdot} \|_{\mathscr{L}(E)}$ is bounded by a constant for $0 \leqslant t \leqslant T$. Hence $d/dt(v_{\lambda,n}) \in L^\infty(0, T; E) \subset L^q(0, T; E')$ and it is clear that $v_{\lambda,n}(0) = 0$. Thus $v_{\lambda,n} \in D(L)$. Now $u * \varphi_n \to u$ in $L^p(0, T; E)$, and so $\lambda M_\lambda(u * \varphi_n) \to \lambda M_\lambda u$ in $W$. Finally, $\lambda M_\lambda(t) \to I$ strongly in $E$ for each $t$. Hence, using (1.2) and the dominated convergence theorem,

$$S\lambda M_\lambda u = \lambda M_\lambda S u \to Su \quad \text{as} \quad \lambda \to \infty \quad \text{in} \quad L^p(0, T; E).$$

That is, $\lambda M_\lambda u \to u$ in $W$. Thus $D(L)$ is dense in $W$, and a similar argument shows $D(L')$ also dense in $W$. Q.E.D.

Lemma 3.1 then permits us to define $L_s$ as the closure of $L$ in $W \times W'$, and $L_w$ as the adjoint of $L'$, taken in $W \times W'$. Evidently, $L_s \subset L_w$.

LEMMA 3.2. *Suppose* $1 < p < \infty$, *and that* (1.1), (1.2), *and* (1.3) *are satisfied so that* $D(L_s)$ *is well-defined. Let* $u \in D(L_s)$. *Then* $u$ *is equal a.e. to a continuous function on* $[0, T]$ *with values in* $E'$ *and*

$$k \mid u(t)|_{E'}^2 \leqslant 2 \operatorname{Re} \int_0^t \left\langle \frac{du}{ds}, u(s) \right\rangle_{V'(s),V(s)} ds \tag{3.2}$$

*for* $0 \leqslant t \leqslant T$.

*Proof.* Suppose first that $u \in D(L)$. Then extend $u$ by 0 for $t < 0$, and by reflection and smooth truncation for $t > T$. Again denoting the extended function by $u$, we have $u \in L^p(R; E)$ with $du/dt \in L^q(R, E')$. Then using the functions $\varphi_n$ of Lemma 3.1, we have that $u_n = u * \varphi_n$ are smooth, $u_n \to u$ in $L^p(R, E)$, and $du_n/dt \to du/dt$ in $L^q(R, E')$. Now clearly for $u_n$ we have

$$k \mid u_n(t)|_{E'}^2 \leqslant \langle u_n(t), u_n(t) \rangle_{E,E'} = 2 \operatorname{Re} \int_0^t \left\langle \frac{du_n}{ds}, u_n(s) \right\rangle_{E',E} ds.$$

But since $u_n \to u$ uniformly on $[0, T]$ in $E'$, we have

$$k \mid u(t)|_{E'}^2 \leqslant 2 \operatorname{Re} \int_0^t \left\langle \frac{du}{ds}, u(s) \right\rangle_{E',E} ds = 2 \operatorname{Re} \int_0^t \left\langle \frac{du}{ds}, u(s) \right\rangle_{V'(s),V(s)} ds,$$

using the identification (3.1). The inequality can then be extended to the closure $D(L_s)$ in the usual manner. Q.E.D.

Next we recall some important definitions. Let $X$ be a reflexive Banach space over $\mathbf{C}$, and let $X'$ denote the antidual with the usual dual norm.

A (possibly nonlinear) operator $B$ with domain $D(B) \subset X$ and range in $X'$ is said to be *monotone* if for all $u, v \in D(B)$,

$$\operatorname{Re}\langle Bu - Bv, u - v \rangle_{X',X} \geqslant 0.$$

A linear monotone operator $T$ with domain $D(T) \subset X$ and range in $X'$ is said to be *maximal monotone* if $T$ has no monotone linear extension.

A (possibly nonlinear) operator $B$ with $D(B) = X$ and range in $X'$ is said to be *bounded* if $B$ takes bounded sets in $X$ into bounded sets in $X'$, *hemicontinuous* if $B$ is continuous on lines in $X$ to weak topology of $X'$, and *coercive* if there is a real-valued function $c(s)$, $s \in R^+$, such that for all $u \in X$,

$$\operatorname{Re}\langle Bu, u \rangle \geqslant c(\| u \|) \, \| u \|_X$$

and $c(s) \to +\infty$ as $s \to \infty$.

Lemma 3.2 implies that $L$ and $L_s$ are both monotone, and a similar argument shows that $L'$ is monotone.

We now state a recent theorem of Brezis [5] and a result of Browder [7] regarding monotone operators.

THEOREM (Brezis). *Let $T$ be a densely defined, monotone, linear operator with domain $D(T)$ in a reflexive Banach space $X$, and range in $X'$. Then $T$ is maximal monotone if and only if $T^*$ is maximal monotone.*

THEOREM (Browder). *Let $T$ be a maximal monotone linear operator with domain $D(T)$ in a reflexive Banach space $X$ and range in $X'$. Let $A : X \to X'$ be a monotone, hemicontinuous, coercive, and bounded operator. Then $T + A$: $D(T) \to X'$ is onto.*

Now suppose that $M$ is a maximal monotone extension of $L'$. Then $M^* \subset L_w$ and $M^*$ is maximal monotone by Brezis' theorem. Combining this with the theorem of Browder and Lemma 3.1 leads to the following.

THEOREM 3.3. *Suppose that* (1.1), (1.2), *and* (1.3) *are satisfied. Let $A : W \to W'$ be monotone, hemicontinuous, coercive, and bounded. Then $L_w + A : D(L_w) \to W'$ is onto.*

Suppose next that $A : W \to W'$ actually has the form

$$(Au)(t) = A(t)\, u(t),$$

where for each $t \in [0, T]$, $A(t) : V(t) \to V'(t)$ is a monotone operator.

LEMMA 3.4. *Assume that* (1.1), (1.2), *and* (1.3) *are satisfied* (*so that $L_s$ is well-defined*). *Then*

$$L_s + A : D(L_s) \to W'$$

*is one-to-one.*

*Proof.* Suppose $u, v \in D(L_s)$ such that

$$L_s u + Au = L_s v + Av, \qquad \text{whence} \quad L_s(u - v) + Au - Av = 0.$$

Since $u$, $v \in D(L_s)$, it follows by Lemma 3.2 that $u - v$ is continuous with values in $E'$, and that for each $t$, $0 \leqslant t \leqslant T$,

$$\begin{aligned} 0 &= 2 \operatorname{Re} \int_0^t \left\langle \frac{du}{ds} - \frac{dv}{ds}, u(s) - v(s) \right\rangle ds \\ &\quad + 2 \operatorname{Re} \int_0^t \langle A(s)\, u(s) - A(s)\, v(s), u(s) - v(s) \rangle\, ds \\ &\geqslant k \mid u(t) - v(t) \mid_{E'}^2 ; \end{aligned}$$

thus $u = v$. Q.E.D.

Thus far, we have assumed only that $1 < p < \infty$. However, to obtain the important equality $L_s = L_w$, we have had to assume $p \geqslant 2$, which we do not feel should be essential. The following theorem gives a criterion for this equality.

THEOREM 3.5. *Assume $p \geqslant 2$ and that* (1.1), (1.2), *and* (1.3) *are satisfied. Then $L_s = L_w$.*

*Proof.* Suppose that $u \in W$ and $f \in W'$ such that

$$-\int_0^T \left\langle u, \frac{dv}{dt} \right\rangle_{E',E} dt = \int_0^T \langle f, v \rangle_{V(t)',V(t)} \, dt \tag{3.3}$$

for all $v \in W$ such that $dv/dt \in L^q(0, T; E)$, $v(T) = 0$. Then to show $L_s = L_w$, it will suffice to show that $u \in D(L_s)$. Now for $v$ in (3.3) we may take

$$v(t) = M_\lambda(t)\, g(t), \qquad \lambda > 0,$$

where $g \in L^p(0, T; E)$, $dg/dt \in L^p(0, T; E)$, $g(T) = 0$. Then

$$\frac{dv}{dt} = M_\lambda{}^{\cdot} g + M_\lambda \frac{dg}{dt} \in L^p(0, T; E).$$

Substituting in (3.3) we obtain

$$-\int_0^T \left\langle u, M_\lambda \frac{dg}{dt} \right\rangle dt = \int_0^T \langle u, M_\lambda{}^{\cdot} g \rangle \, dt + \int_0^T \langle f, M_\lambda g \rangle \, dt. \tag{3.4}$$

The last term on the right may be written

$$\begin{aligned} \int_0^T \langle f, M_\lambda g \rangle \, dt &= \int_0^T \langle (S^{-1})^\wedge f, S M_\lambda g \rangle \, dt \\ &= \int_0^T \langle (S M_\lambda)^* (S^{-1})^\wedge f, g \rangle \, dt, \end{aligned}$$

where the duality in the latter two expressions is between $E'$ and $E$. Using the symmetry of $M_\lambda(t)$ and $M_\lambda(t)^{\cdot}$, (3.4) becomes

$$\begin{aligned} -\int_0^T \left\langle M_\lambda u, \frac{dg}{dt} \right\rangle dt = &\int_0^T \langle M_\lambda{}^{\cdot} u, g \rangle \, dt \\ &+ \int_0^T \langle (S M_\lambda)^* (S^{-1})^\wedge f, g \rangle \, dt, \end{aligned}$$

where all the dualities are between $E$ and $E'$. It follows that in $\mathscr{D}'(E')$ we have

$$\frac{d}{dt}(M_\lambda u) = M_\lambda{}^{\cdot} u + (S M_\lambda)^* (S^{-1})^\wedge f,$$

and the right side is in $L^q(0, T; E')$. To show that $(M_\lambda u)(0) = 0$, so that $M_\lambda u \in D(L)$, we choose $\psi \in C_0^\infty(-\infty, T; E)$ so that $M_\lambda \psi \in D(L')$, and substitute again in (3.3). Integrating the right side by parts ($M_\lambda u$ is continuous with values in $E'$), we obtain

$$\langle (M_\lambda u)(0), \psi(0) \rangle_{E',E} = 0,$$

which implies that $(M_\lambda u)(0) = 0$ since $E$ is dense in $E'$.

We set $u_\lambda = \lambda M_\lambda \in D(L)$. We have seen in Lemma 3.1 that

$$Su_\lambda = \lambda M_\lambda Su \to Su$$

in $L^p(0, T; E)$ as $\lambda \to +\infty$ so that $u_\lambda \to u$ in $W$ as $\lambda \to +\infty$. It remains to show that $du_\lambda/dt \to f$ in $W'$, or equivalently, that

$$S^{-1} \frac{du_\lambda}{dt} \to (S^{-1})\hat{}f \qquad \text{in} \quad L^q(0, T; E').$$

Now

$$S^{-1} \frac{du_\lambda}{dt} = \lambda S^{-1} M_\lambda{}^{\cdot} u + \lambda S^{-1}(SM_\lambda)^* \, (S^{-1})\hat{}f.$$

By Lemma 2.4, we know that the first term converges to zero in $L^p(0, T; E)$, and hence in $L^q(0, T; E')$. As for the second term, we note that $(S^{-1})\hat{}f \in L^q(0, T; E')$, and that because of the symmetry hypothesis (1.1) and Lemma 2.1,

$$\begin{aligned} S^{-1}(t)(S(t)\, M_\lambda(t))^* &= S^{-1}(t)^*(S(t)\, M_\lambda(t))^* \\ &= M_\lambda(t)^*. \end{aligned}$$

Thus the second term is $\lambda M_\lambda{}^*(S^{-1})\hat{}f$. Now for $g \in E$, $\lambda M_\lambda{}^* g = \lambda M_\lambda g \to g$ in $E$ as $\lambda \to +\infty$. But $\| \lambda M_\lambda{}^* \|_{\mathscr{L}(E')} = \| \lambda M_\lambda \|_{\mathscr{L}(E)}$ and $E$ is dense in $E'$, with continuous injection. Hence $\lambda M_\lambda{}^*(S^{-1})\hat{}f \to (S^{-1})\hat{}f$ in $L^q(0, T; E')$.

Q.E.D.

COROLLARY. *Assume $p \geqslant 2$ and let* (1.1), (1.2), *and* (1.3) *be satisfied. Suppose that $A : W \to W'$ is hemicontinuous, bounded, and coercive, and that for each $t \in [0, T]$, $(Au)(t) = A(t)\, u(t)$, where $A(t) : V(t) \to V'(t)$ is monotone. Then*

$$L_s + A : D(L_s) \to W'$$

*is one-to-one and onto.*

*Remark.* If instead of $S^2(t)$ we assume that $S(t)$ satisfies (1.1), (1.2), and (1.3), one may prove Theorems 3.3 and 3.5 using very similar arguments. However, it will usually be more difficult in practice to check these latter hypotheses, and one can give examples (see Ref. [10]) where $S(t)$ does not satisfy (1.3) while $S^2(t)$ does.

Finally we should mention the work of Baiocchi [3], in which he also considers a variable-domain problem, but relies heavily on certain Hilbert space interpolation results. Our method, which as this paper shows, can be extended to a Banach space situation, does not require that there exist a constant subspace $V \subset V(t)$, all $t$, with $V$ dense, as is the case with the paper of Baiocchi.

**4.** We begin this section of examples by noting that if $E$ is a Hilbert space, then one can make the identification $E = E'$. If $V(t)$, $0 \leqslant t \leqslant T$, is a family of dense subspaces of $E$, which are also Hilbert spaces such that the injection $V(t) \to E$ is continuous, then it is well-known that there is a self-adjoint, invertible, positive operator $S^2(t)$ in $E$ such that $D(S(t)) = V(t)$ and the scalar product on $V(t)$ can be written

$$((u, v))_t = (S(t)u, S(t)v), \qquad u, v \in V(t),$$

where $(\cdot, \cdot)$ is the scalar product of $E$. This is the situation dealt with in Ref. [10].

In the case of Banach spaces, however, we do not know of such a general construction and so we have limited our discussion to those subspaces $V(t) \subset E$ such that an operator $S^2(t)$ satisfying (1.1), (1.2), and (1.3) exists. We shall therefore exhibit a class of examples of elliptic operators $\Lambda_p(t)$ in the $L^p$ spaces, $p \geqslant 2$, which satisfy (1.1), (1.2), and (1.3). Then one can define a square root of these operators and write $\Lambda_p(t) = S^2(t)$; finally, we set $V(t) = D(S(t)) = D(\Lambda_p^{1/2}(t))$.

To begin, we let $\Omega$ be a bounded, open set in $R^n$ lying on one side of its boundary $\Gamma$ which we assume to be an $(n - 1)$-dimensional $C^\infty$ manifold. Let $L^p(\Omega)$ be defined as usual with the Lebesgue measure of $R^n$. For $f \in L^p(\Omega)$ we shall write the norm as $|f|_p$ without reference to $\Omega$ unless it becomes necessary to avoid confusion. For $p \geqslant 2$ we have

$$L^p(\Omega) \subset L^q(\Omega) = [L^p(\Omega)]'$$

if $1/p + 1/q = 1$. Thus with $E = L^p(\Omega)$, $p \geqslant 2$, we have satisfied the first requirement of our theory. (Of course, the "$p$" used here need not be the same as that in Sections 1, 2, 3.)

We shall denote points $x \in R^n$ by $x = (x_1, \ldots, x_n)$, and if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an $n$-tuple of integers $\alpha_j \geqslant 0$, we set

$$D^\alpha u(x) = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} u,$$

where $D_k^{\alpha_k} = \partial^k u(x)/\partial x_k^{\alpha_k}$.

The Sobolev spaces $W^{m,p}(\Omega)$ are defined, for $m \geqslant 0$ an integer, as the space of distributions on $\Omega$ such that

$$D^\alpha u \in L^p(\Omega), \qquad |\alpha| \leqslant m, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n .$$

$W^{m,p}(\Omega)$ is a Banach space (reflexive for $p > 1$) with the norm

$$\| u \|_{m,p} = \left[ \int_\Omega \sum_{|\alpha| \leqslant m} | D^\alpha u |^p \, dx \right]^{1/p}.$$

Let $\mathscr{D}(\bar{\Omega})$ denote the space of functions which are $C^\infty$ on the closure of $\Omega$. For $u \in \mathscr{D}(\bar{\Omega})$, $\gamma_j u$, $j \geqslant 0$ an integer, is defined to be the $j$-th order derivative of $u$ in the exterior normal direction at the boundary.

It is well-known that for $p > 1$, the mapping $u \to \gamma_j u$ $(0 \leqslant j \leqslant m - 1)$ extends as a bounded linear operator

$$\gamma_j : W^{m,p}(\Omega) \to W^{m-j-1/p,p}(\Gamma).$$

Furthermore, if $\{B_j(D)\}$, $j = 1,..., m$, is a normal family of boundary operators with the order $m_j$ of $B_j \leqslant m - 1$, then the mapping

$$(B_1 ,..., B_m) : W^{m,p}(\Omega) \to \prod_{j=0}^{m-1} W^{m-j-1/p,p}(\Gamma)$$

is surjective, and there is a continuous "lifting"

$$\phi : \prod_{j=0}^{m-1} W^{m-j-1/p,p}(\Gamma) \to W^{m,p}(\Omega)$$

such that $(B_1 ,..., B_m) \circ \phi =$ identity in $\prod_{j=0}^{m-1} W^{m-j-1/p,p}(\Gamma)$. For definitions of $W^{m-j-1/p,p}(\Gamma)$ and proofs we refer the reader to Refs. [4] or [16].

Now let $k_0$ be a positive integer, $0 \leqslant k_0 < m - 1$. We set $K = \{0,..., k_0\}$ and let $J$ be a subset of $\{k_0 + 1,..., m - 1\}$. Now consider the boundary operators

$$\gamma_j - \sum_K a_{jk}(t) \gamma_k , \qquad j \in J,$$

where $a_{jk}(t)$ is a differential expression of order $\leqslant j - k$ in the tangential derivatives and $a'_{jk}(t)$ is its formal adjoint (but without conjugating the coefficients). We assume

(4.1) The coefficients of $a_{jk}(t)$ and $a'_{jk}(t)$ are of class $C^{2m}$ for $x \in \Gamma$, and of class $C^1$ on $\Gamma \times [0, T]$.

Now define a closed subspace $K(t)$ of $H^m(\Omega) = W^{m,2}(\Omega)$ as the space of functions $u \in H^m(\Omega)$ such that

$$\gamma_j u = \sum_K a_{jk}(t)\, \gamma_k u, \qquad j \in J. \tag{4.2}$$

We define a continuous sesquilinear form on $H^m(\Omega)$,

$$a(u, v) = \int_\Omega \sum_{k=1}^{m} D_k{}^m u D_k{}^m \bar{v}\, dx + c_0 \int_\Omega u\bar{v}\, dx,$$

with $c_0 > 0$ to be chosen later. $a(u, v)$ is Hermitian, i.e., $a(u, v) = a(\overline{v, u})$. Furthermore, by Agmon [1], we know that for $c_0$ sufficiently large,

$$a(u, u) \geqslant c \| u \|^2_{H^m(\Omega)} \qquad \text{for all} \quad u \in H^m(\Omega),$$

where $c$ is a positive constant. Hence the form $a(u, v)$ and the space $K(t)$ give rise to an unbounded self-adjoint operator in $L^2(\Omega)$ which we shall call $\Lambda_2(t)$.

The domain $D(\Lambda_2(t))$ is the dense subspace of $u \in K(t)$ such that

$$v \to a(u, v)$$

is continuous on $K(t)$ in the topology of $L^2(\Omega)$, and for $u \in D(\Lambda_2(t))$,

$$a(u, v) = (\Lambda_2(t)\, u, v)_{L^2(\Omega)}$$

for all $v \in K(t)$. $\Lambda_2(t)$ is a differential operator with formal differential expression

$$\Lambda u = (-1)^m \sum_{k=1}^{n} D_k^{2m} u + c_0 u.$$

If $u \in D(\Lambda_2(t))$, then $u$ must satisfy certain higher-order boundary conditions in addition to (4.2). They are determined (formally) by the Green's formula

$$(\Lambda u, v) = a(u, v) + \sum_{j=0}^{m-1} \int S_j u \gamma_j \bar{v}\, ds,$$

where $S_j$ is a differential operator of order $2m - 1 - j$.

It can be deduced, using the surjectivity of the trace map, that a smooth function $u \in D(\Lambda_2(t))$ satisfies

$$\begin{aligned} \gamma_j u - \sum_{k \in K} a_{jk}(t)\, \gamma_k u &= 0, \qquad j \in J, \\ S_j u + \sum_{k \in J} a'_{kj}(t)\, S_k u &= 0, \qquad j \in K, \\ S_j u &= 0, \qquad j \notin K \cup J, \end{aligned} \tag{4.3}$$

where $a'_{kj}(t)$ is the formal adjoint of $a_{jk}(t)$ (but without conjugating the coefficients).

Thus we have a total of $m$ boundary conditions which determine the domain of $A_2(t)$. We shall relabel the boundary operators (4.3)

$$B_j(t), \qquad j = 0,..., m-1,$$

where $B_j(t)$ has order $m_j$ and $0 \leqslant m_j \leqslant 2m - 1$.

Next we define $A_p(t)$ in $L^p(\Omega)$ ($p > 1$) as the differential operator

(4.4) $A_p(t)u = Au$ with $D(A_p(t))$ the space

$$\{u \in W^{2m,p}(\Omega) : B_j(t)u = 0,\ j = 0,..., m-1\}$$

The differential expression $Au = (-1)^m \sum_{k=1}^n D_k^{2m} u + c_0 u$ is uniformly strongly elliptic and the boundary conditions $B_j(t)$ given by (4.3) are normal (cf. Ref. [4]). Furthermore, a simple modification of the argument on p. 680 of Agmon, Douglis, and Nirenberg, Ref. [2], shows that because the form $a(u, v)$ is coercive on $H^m(\Omega)$, the $B_j$ also satisfy the complementing condition (covering property) with respect to $A$. Thus $A$ and the boundary conditions $B_j(t)$ constitute a family of regular elliptic boundary value problems.

We refer the reader to Browder [6] for proof of the following a priori estimate.

Let $A$ be a strongly elliptic operator in $\Omega$ with constant coefficients, and $B_j(t)$, $j = 0,..., m-1$, a family of boundary operators with order $m_j < 2m$ which is normal and satisfies the complementing condition. If $u \in W^{2m,p}(\Omega)$, $1 < p < \infty$, and $B_j(t)u = 0$, $j = 0,..., m-1$, then $Au \in L^p(\Omega)$ and there is a constant $c_p$, independent of $u$, such that

$$\| u \|_{2m,p} \leqslant c_p\{| Au |_p + | u |_p\}. \tag{4.5}$$

Furthermore, the constant $c_p$ depends only on $p$, $\Omega$, and on a bound over $\Gamma$ for the coefficients in the boundary operators $B_j(t)$, $0 \leqslant j \leqslant m-1$.

Lemma 4.1. *Suppose $A_p(t)$ is given as in* (4.4) *and that* (4.1) *is satisfied. Then there are constants $c_1$ and $c_2 > 0$, independent of $t \in [0, T]$ such that for* $\operatorname{Re} \lambda \geqslant \lambda_0 > 0$ *sufficiently large, and* $|\arg \lambda| \leqslant \pi/2 - \epsilon$, $\epsilon > 0$,

$$\begin{aligned} \| u \|_{2m,p} &\leqslant c_1 \,| A_p(t)\, u + \lambda u |_p\,, \\ | u |_p \,| \lambda | &\leqslant c_2 \,| A_p(t)\, u + \lambda u |_p\,. \end{aligned} \tag{4.6}$$

*Proof.* The proof is a simple modification of that given on p. 70 of Friedman, Ref. [11], for the Dirichlet problem. We let $\Omega_k = \Omega \times \{| s | \leqslant k\}$ for $k = 0, 1, 2,...$ . Since we are assuming $\Omega$ has a $C^\infty$ boundary, we can construct a domain $\Omega^*$, also with a $C^\infty$ boundary, such that $\Omega_2 \subset \Omega^* \subset \Omega_3$. Next we introduce the differential operator

$$A' = A + (-1)^m e^{i\theta} D_s^{2m} \qquad (-\tfrac{1}{2}\pi + \epsilon \leqslant \theta \leqslant \tfrac{1}{2}\pi - \epsilon).$$

Let $\chi(s)$ be a $C^\infty$ function such that $\chi(s) = 1$ for $|s| \leqslant 1$, and $\chi(s) = 0$ for $|s| \geqslant 3/2$. Then we extend the boundary operators $B_j(t)$ to the boundary of $\Omega^*$ as

$$\begin{aligned} B_j'(t) &= \gamma_j' - \sum_K \chi(s)\, a_{jk}(t)\, \gamma_k, && j \in J, \\ B_j'(t) &= S_j' - \sum_{k \in J} \chi(s)\, a'_{kj}(t)\, S_k, && j \in K, \\ B'_j(t) &= S_j', && j \notin J \cup K, \end{aligned} \tag{4.7}$$

where $\gamma_j'$ is the $j$-th exterior normal derivative on $\partial\Omega^*$ and $S_j'$ is the differential operator of order $2m - j - 1$ resulting from the Green's formula and $A'$. It is clear that the new operator $A'$ and the boundary operators (4.7) again constitute a family of regular elliptic boundary value problems satisfying (4.1) on $[0, T]$. Hence the a priori estimates of Browder apply in $\Omega^*$ with a constant $c_p'$ independent of $t \in [0, T]$. Thus if $u \in D(A_p(t))$, then

$$v(s, x) = u(x)\, \chi(s)\, e^{i\mu s}, \quad \mu \text{ real},$$

belongs to $W^{2m,p}(\Omega^*)$ and satisfies $B_j'(t)v = 0$ for $j = 1,\ldots, m$, whence

$$\| v \|_{2m,p}^{\Omega_*} \leqslant c_p' \{ | A_p'(t)\, v |_p^{\Omega_*} + | v |_p^{\Omega_*} \},$$

where the constant $c_p'$ is again independent of $t \in [0, T]$.

Now

$$\begin{aligned} \| u e^{i\mu t} \|_{2m,p}^{\Omega_1} \leqslant{} & c\, | Au + \mu^{2m} e^{i\theta} u |_p^{\Omega} \\ & + c\, |\mu|^{2m-1}\, | u |_p^{\Omega} + c\, | u |_p^{\Omega}, \end{aligned}$$

where $c$ denotes various constants independent of $t$. On the other hand,

$$\begin{aligned} [\| u e^{i\mu t} \|_{2m,p}^{\Omega_1}]^p &\geqslant 2 \sum_{j=0}^{2m} \int_\Omega \sum_{|\alpha| \leqslant j} | \mu^{2m-j} D^\alpha u |^p \, dx \\ &\geqslant 2(\mu \| u \|_{k,p}^{\Omega})^p \qquad \text{for any} \quad k = 0,\ldots, 2m. \end{aligned}$$

Then taking $\lambda = \mu^{2m} e^{i\theta}$ and $|\mu| \geqslant \mu_0$ sufficiently large, we obtain (4.7).

Q.E.D.

We may assume, therefore, that $c_0 > 0$ is chosen so large that $A_p(t)$ is invertible in $L^p(\Omega)$ for each $t$, and that $A_p(t)$ satisfies (1.2). If we assume that $p \geqslant 2$, so that

$$L^p(\Omega) \subset L^q(\Omega), \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

it is clear that $A_p(t)$ also satisfies (1.1). It remains to show that (1.3) is satisfied. The following argument is adapted from Lions, Ref. [15], Chapter VII.

LEMMA 4.2. *Let $f \in L^p(\Omega)$ and suppose that* (4.1) *is satisfied. Then $A_p^{-1}(t) f$ is continuous on* $[0, T]$ *with values in* $W^{2m,p}(\Omega)$.

*Proof.* Suppose for the moment that for any $v \in D(A_p(s))$, $0 \leqslant s \leqslant T$, we may find a function $g(t)$ with values in $W^{2m,p}(\Omega)$ such that

(i) $v - g(t) \in D(A_p(t))$;

(ii) $g(t) \to 0$ in $W^{2m,p}(\Omega)$ as $t \to s$.

Then choose $g(t)$ such that $A_p^{-1}(s) f - g(t) \in D(A_p(t))$. Using (4.6) we have (and denoting the norm in $W^{2m,p}$ by $\| \cdot \|$)

$$\begin{aligned} \| A_p^{-1}(t) f - A_p^{-1}(s) f \| &\leqslant \| A_p^{-1}(t) f - A_p^{-1}(s) f + g(t) \| + \| g(t) \| \\ &\leqslant c_1 \, | A(A_p^{-1}(t) f - A_p^{-1}(s) f + g(t)) | + \| g(t) \| \\ &\leqslant (c_1 + 1) \| g(t) \|, \end{aligned}$$

so that $\| A_p^{-1}(t) f - A_p^{-1}(s) f \| \to 0$ as $t \to s$.

Now let us return to the proof of (i) and (ii). For $v \in D(A_p(s))$ we wish to find $g(t)$ satisfying (i), i.e.,

$$\begin{aligned} \gamma_j g(t) - \sum_{k \in K} a_{jk}(t) \, \gamma_k g(t) &= \gamma_j v - \sum_{k \in K} a_{jk}(t) \, \gamma_k v, & j &\in J, \\ S_j g(t) + \sum_{k \in J} a'_{kj}(t) \, S_k g(t) &= S_j v - \sum_{k \in J} a_{jk}(t) \, S_k v, & j &\in K, \\ S_j g(t) &= S_j v = 0, & j &\notin J \cup K. \end{aligned}$$

This will be the case if

$$\begin{aligned} \gamma_j g(t) &= 0, & j &\notin J, \\ \gamma_j g(t) &= \gamma_j v - \sum_{k \in K} a_{jk}(t) \, \gamma_k v, & j &\in J, \\ S_j g(t) &= 0, & j &\notin K, \\ S_j g(t) &= S_j v + \sum_{k \in K} a'_{kj}(t) \, S_k v, & j &\in K. \end{aligned} \tag{4.8}$$

Since the boundary operators $\{\gamma_j, S_j\}$, $j = 0, \ldots, m-1$, form a normal family, and the right sides of (4.8) lie in the proper spaces, we know there is an element $g(t) \in W^{2m,p}(\Omega)$ satisfying the equations (4.8), and furthermore, that

$$\| g(t) \|_{W^{2m,p}(\Omega)} \leqslant \sum_{j \in J} \| \gamma_j v - \sum_{k \in K} a_{jk}(t)\, \gamma_k v \|_{W^{2m-j-1/p,p}(\Gamma)} + \sum_{j \in K} \| S_j v + \sum_{k \in J} a'_{kj}(t)\, S_k v \|_{W^{j+1-1/p,p}(\Gamma)} .$$

Now because $v \in D(A_p(s))$ and the coefficients in $a_{jk}(t)$ and $a'_{kj}(t)$ are continuous in $t$, we must have

$$\gamma_j v - \sum_{k \in K} a_{jk}(t)\, \gamma_k v \to 0 \qquad \text{in} \quad W^{2m-j-1/p,p}(\Gamma), \qquad j \in J,$$

and

$$S_j v + \sum_{k \in J} \bar{a}_{kj} S_k v \to 0 \qquad \text{in} \quad W^{j+1-1/p,p}(\Gamma), \qquad j \in K,$$

as $t \to s$. Thus (ii) is satisfied and the lemma is proved.

LEMMA 4.3. *Let $f \in L^p(\Omega)$ and suppose that* (4.1) *holds. Then $A_p^{-1}(t) f$ is continuously differentiable in $W^{2m,p}(\Omega)$ on* $[0, T]$.

*Proof.* Let us set $u(t) = A_p^{-1}(t) f$; then suppose for the moment that we can find a function $w_h(t) \in W^{2m,p}(\Omega)$ for small $h$ real such that

(i) $u(t+h) - w_h(t) \in D(A_p(t))$;

(ii) $w_h(t)/h \to w'(t)$ in $W^{2m,p}(\Omega)$ as $h \to 0$;

(iii) $t \to w'(t)$ is continuous in $W^{2m,p}(\Omega)$ on $[0, T]$.

Then $[u(t+h) - u(t)]/h \to -A_p^{-1}(t)\, A w'(t) + w'(t)$ in $W^{2m,p}(\Omega)$. In fact,

$$u(t+h) - u(t) = u(t+h) - w_h(t) - u(t) + w_h(t)$$

and

$$\left\| \frac{1}{h} [u(t+h) - w_h(t) - u(t)] + A_p^{-1}(t)\, A w'(t) \right\| \leqslant c_1 \left| A \frac{w_h(t)}{h} - A w'(t) \right|_{L^p(\Omega)} \leqslant c_1 \left\| \frac{w_h(t)}{h} - w'(t) \right\| \to 0 \qquad \text{as} \quad h \to 0,$$

where we have used (4.6) again. Since $w'(t)$ is assumed strongly continuous in $W^{2m,p}(\Omega)$ it follows that $u(t)$ is continuously differentiable in $W^{2m,p}(\Omega)$ on $[0, T]$. Thus the lemma is proved, modulo the function $w_h(t)$ verifying (i), (ii), (iii).

$w_h(t)$ satisfying (i) means

$$\gamma_j w_h(t) - \sum_{k\in K} a_{jk}(t)\, \gamma_k w_h(t) = \gamma_j u(t+h) - \sum_{k\in K} a_{jk}(t)\, \gamma_k u(t+h), \qquad j \in J,$$

$$S_j w_h(t) - \sum_{k\in J} a'_{kj}(t)\, S_k w_h(t) = S_j u(t+h) - \sum_{k\in J} a'_{kj}(t)\, S_k u(t+h), \qquad j \in K,$$

$$S_j w_h(t) = 0, \qquad j \notin J \cup K.$$

Since $u(t+h) \in D(A_p(t+h))$, these equations will be satisfied if

$$\begin{aligned} \gamma_j w_h(t) &= 0, && j \notin K,\\ \gamma_j w_h(t) &= \sum_K [a_{jk}(t+h) - a_{jk}(t)]\, \gamma_k u(t+h), && j \in J,\\ S_j w_h(t) &= \sum_J [a'_{kj}(t+h) - a'_{kj}(t)]\, S_k u(t+h), && j \in K,\\ S_j w_h(t) &= 0, && j \notin K. \end{aligned}$$

Again using the fact that $\{\gamma_j, S_j\}$, $j = 0,..., m-1$, is a normal family of boundary operators, we know that for each $h$ and $t$ a solution exists in $W^{2m,p}(\Omega)$ which will depend continuously on the right side of the equations above. Thus

$$\begin{aligned} \gamma_j w_h(t) &= \varphi_j(t, h), && j = 0,..., m-1,\\ S_j w_h(t) &= \psi_j(t, h), && j = 0,..., m-1, \end{aligned}$$

where by (4.1) and Lemma 4.2,

$$\frac{\varphi_j(t, h)}{h} \to \varphi_j'(t) \qquad \text{in} \quad W^{2m-j-1/p,p}(\Gamma)$$

and

$$\frac{\psi_j(t, h)}{h} \to \psi_j'(t) \qquad \text{in} \quad W^{j+1-1/p,p}(\Gamma).$$

$\varphi_j'(t)$ and $\psi_j'(t)$ are continuous on $[0, T]$ in the spaces indicated above. By the continuity of the lifting $\phi$, it follows that $w_h(t)$ satisfies (ii) and (iii). Q.E.D.

Thus we have shown that for $p > 1$, $A_p(t)$ satisfies (1.2) and (1.3), and that for $p \geqslant 2$, also satisfies (1.1). Since (1.2) holds, we may take the square root of $A_p(t)$, we may write

$$A_p(t) = S^2(t),$$

where

$$S^{-1}(t)f = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(A_p(t) + \lambda)^{-1}\, d\lambda.$$

The theorems of Section 3 apply then to the subspaces

$$V(t) = D(A_p^{1/2}(t))$$

equipped with the graph norm.

*Remarks.* For $p = 2$,

$$\begin{aligned} D(A_2^{1/2}(t)) &= K(t) \\ &= \Big\{u \in H^m(\Omega) : \gamma_j u = \sum_{k\in K} a_{jk}(t)\,\gamma_k u,\, j \in J\Big\} \end{aligned}$$

(cf. Grisvard [12]). This justifies our only having considered

$$A = (-1)^m \sum_{i=1}^{n} D_i^{2m} + c_0 I$$

in the discussion of this section. We sought the simplest elliptic differential operator which would be self-adjoint on a domain determined in part by the lower-order (i.e., $\leqslant m - 1$) boundary conditions

$$\gamma_j u = \sum_{k\in K} a_{jk}(t)\,\gamma_k u = 0, \qquad j \in J.$$

We were thus led to constant coefficients for $A$ and the variational form on $H^m(\Omega)$.

Unfortunately, it is more difficult to identify the domain of $A^{1/2}(t)$ in the case $p \neq 2$. One cannot, in general, identify the domain of a fractional power $A^\alpha$ with real interpolation spaces between $D(A)$ and $L^p(\Omega)$ (cf. Komatsu [14]).

*Added in proof.* The recent work of R. Seeley show that the complex method of interpolation can be used to identify the domain of $A_p^{1/2}$, $p \neq 2$, with results entirely analogous to those of Grisvard cited above.

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